MOTION OF AN EVAPORATING DROPLET

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UDC 532.58:536.423.16

The steady-state velocity field of a vapor is calculated and also the resistance force of a droplet in motion. This force is found to decrease appreciably as the evaporation rate increases.

We consider a droplet in a vapor stream with the velocity u at infinity. It is assumed that the droplet evaporates into a radial spherically symmetrical vapor stream with the velocity w at the droplet surface, so as to simultaneously satisfy the conditions $ua \ll \nu$ and $uwa^2 \ll \nu^2$ (a denoting the droplet radius ν denoting the kinematic viscosity). The large difference between liquid and vapor density, as long as the pressure and the temperature remain far from critical, allow us to disregard the change in the droplet radius within the region of the order of $a(1 + wa/\nu)$ while the velocity field around the droplet becomes steady.

The flow of vapor around a droplet is described by the Navier–Stokes equation and the continuity equation, which are well known [1] to yield an equation for the flow function:

$$\frac{R}{r^2 \sin\theta}\left(\frac{\partial \Psi}{\partial \theta}\cdot\frac{\partial}{\partial r} - \frac{\partial \Psi}{\partial r}\cdot\frac{\partial}{\partial \theta} + 2\mathrm{ctg}\,\theta\,\frac{\partial \Psi}{\partial r} - \frac{2}{r}\cdot\frac{\partial \Psi}{\partial \theta}\right) D^2\Psi = D^4\Psi,$$

where

$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\cdot\frac{\partial}{\partial \theta}\cdot\frac{1}{\sin\theta}\cdot\frac{\partial}{\partial \theta},\ R = \frac{wa}{\nu},$$
$$v_r = \frac{1}{r^2\sin\theta}\cdot\frac{\partial \Psi}{\partial \theta},\quad v_\theta = -\frac{1}{r\sin\theta}\cdot\frac{\partial \Psi}{\partial r}. \tag{1}$$

Here Ψ is a dimensionless flow function, r is the dimensionless distance from the center of the droplet, and v_r, v_θ are components of the dimensionless velocity in spherical coordinates with the polar axis in the direction of the stream velocity at infinity **u**. In order to change to ordinary dimensional quantities, it is necessary to replace r, v_r, v_θ, and Ψ by r/a, v_r/w, v_θ/w, and Ψ/wa^2 respectively.

The boundary conditions are

$$\Psi = -\cos\theta,\ \frac{\partial \Psi}{\partial r} = 0 \ \text{at}\ r = 1,$$
$$\Psi \to \frac{1}{2}\varepsilon r^2 \sin^2\theta,\ \text{where}\ \varepsilon = \frac{u}{w}\ \text{at}\ r \to \infty. \tag{2}$$

Equation (1) and the boundary conditions (2) allow us to seek the flow function in the form

$$\Psi(r,\ \theta) = -\cos\theta + \varepsilon\psi(r)\sin^2\theta. \tag{3}$$

Then accurately within ε-order terms, (1) and (2) yield

$$\frac{R}{r^2}\left(\frac{d}{dr} - \frac{2}{r}\right)\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)\psi = \left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)^2\psi,$$
$$\psi = 0,\ \frac{d\psi}{dr} = 0\ \text{at}\ r = 1,\ \psi \to \frac{1}{2}r^2\ \text{at}\ r \to \infty. \tag{4}$$

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 24, No. 2, pp. 250-255, February, 1973. Original article submitted September 6, 1971.

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From Eq. (4) we can obtain

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)\psi = Ar^2 E\left(\frac{r}{R}\right), \qquad E(x) = 2 - \left(2 + \frac{2}{x} + \frac{1}{x^2}\right)\exp\left(-\frac{1}{x}\right). \tag{5}$$

Here A = const. The other linearly independent solution has been discarded, because it leads to $\Psi \to r^4$ at $r \to \infty$.

Thus, the solution to Eq. (5) with the corresponding boundary conditions becomes

$$\psi(r) = \frac{1}{2} r^2 \left[F\left(\frac{r}{R}\right) - \frac{R^3}{r^3} G\left(\frac{r}{R}\right)\right],$$

where

$$F(z) \int_{1/R}^{\infty} xE(x)\,dx = \int_{1/R}^{z} xE(x)\,dx, \qquad G(z) \int_{1/R}^{\infty} xE(x)\,dx = \int_{1/R}^{z} x^4 E(x)\,dx. \tag{6}$$

When $R \ll 1$, then function ψ becomes identical to the Stokes solution [2]

$$\psi(r) = \frac{r^2}{2}\left(1 - \frac{3}{2r} + \frac{1}{2r^3}\right) - \frac{9Rr}{32}\left(1 - \frac{2}{r} + \frac{1}{r^2}\right) \tag{7}$$

accurately within R-order terms.

When $R \gg 1$, then disregarding small $1/R^2$-order and e^{-R}-order terms will yield

$$\psi(r) = \frac{r^2}{R^2}\left\{r^2 - 1 - r^2\left(1 + \frac{R}{r}\right)\exp\left(-\frac{R}{r}\right) - \frac{1}{10}\frac{R^3}{r^3} \times E_1\left(\frac{R}{r}\right) - \frac{2}{5}\left[r^2 - \frac{1}{r^3} - r^2\left(1 + \frac{R}{r} + \frac{R^2}{2r^2} - \frac{R^3}{4r^3} + \frac{R^4}{4r^4}\right)\exp\left(-\frac{R}{r}\right)\right]\right\}, \tag{8}$$

where

$$E_1(x) = \int_1^{\infty} \exp(-xt)\frac{dt}{t}.$$

From (8) for $r \gg R$ follows that

$$\psi(r) = \frac{1}{2} r^2 \left(1 - \frac{R}{r}\right). \tag{9}$$

The flow function $\Psi(r, \theta)$ according to (3) and (9) represents a superposition of the velocity field due to a point source on the velocity field of a viscous medium flowing around a sphere of radius Ra.

During a slow evaporation of the droplet ($R \ll 1$), a transition to the velocity of a homogeneous stream occurs within the region $r \sim 1$, but during a fast evaporation of the droplet ($R \gg 1$) such a transition occurs within a much larger region $r \sim R$ and this causes a reduction in the force of viscous friction.

Equation (4) follows from Eq. (1), if one disregards terms of the order $\varepsilon^2\psi^2 \ll 1$. In order that the same boundary condition at $r \to \infty$ for solving Eq. (1) could also be used for solving Eq. (4), it is necessary to require that the condition $\varepsilon r^2 \ll 1$ be satisfied in the region of transition to a homogeneous stream. At a fast evaporation ($R \gg 1$) with appreciable inertia terms in Eq. (1), therefore, it is necessary that the condition $\varepsilon R^2 \ll 1$ be satisfied.

If not, then the transition to the extreme mode of velocity distribution will occur in the region where an incomplete consideration of inertia terms in Eq. (1) may not be valid.

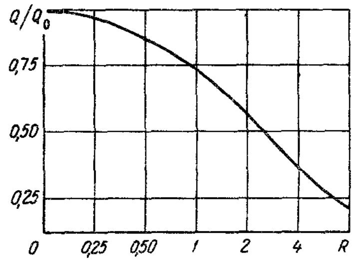

Fig. 1. Viscous resistance force as a function of the evaporation rate, for a droplet in a vapor stream.

Thus, we have

$$v_r = \frac{1}{r^2} + \varepsilon\left[F\left(\frac{r}{R}\right) - \frac{R^3}{r^3}G\left(\frac{r}{R}\right)\right]\cos\theta,$$

$$v_\theta = -\varepsilon\left[F\left(\frac{r}{R}\right) + \frac{R^3}{2r^3}G\left(\frac{r}{R}\right)\right]\sin\theta. \qquad (10)$$

The Navier–Stokes equations with respect to a known velocity field (10) yield the pressure field p of a medium with a density ρ and a pressure p_0 at infinity

$$\frac{p-p_0}{\rho w^2} = -\frac{1}{2r^4} - \frac{\varepsilon}{r^2}\left[F\left(\frac{r}{R}\right) - \frac{R^3}{r^3}G\left(\frac{r}{R}\right) + \frac{3}{2}H\left(\frac{r}{R}\right)\right]\cos\theta,$$

$$H(x)\int_{1/R}^{\infty} xE(x)\,dx = x^2(1-2x)E(x) - x^4\frac{dE(x)}{dx}. \qquad (11)$$

With the aid of formulas (10) and (11), we can now calculate the resistance force of a droplet in a stream. During spherically symmetrical evaporation, obviously, the resistance force Q acts in the direction of the stream velocity u. The projection of this force Q on that velocity u is

$$Q = 2\pi a^2\int_0^{\pi}(-p\cos\theta + \sigma'_{rr}\cos\theta - \sigma'_{r\theta}\sin\theta)\sin\theta\,d\theta,$$

where (12)

$$\sigma'_{rr} = \frac{2\rho w^2}{R}\cdot\frac{\partial v_r}{\partial r}, \quad \sigma'_{r\theta} = \frac{\rho w^2}{R}\cdot\frac{\partial v_\theta}{\partial r}.$$

Consequently,

$$Q = 8\pi\rho\nu a u\,\frac{R\,[1-(1+R)e^{-R}]}{2(1+R)e^{-R}+R^2-2}. \qquad (13)$$

In the case of a slowly evaporating droplet ($R \ll 1$),

$$Q = \left(1 - \frac{7}{24}R\right)Q_0, \qquad (14)$$

where $Q_0 = 6\pi\rho\nu a u$.

The first term of this expression is identical to the Stokes formula [2] for the resistance of a sphere in a viscous stream.

In the case of a fast evaporating droplet,

$$Q = \frac{4}{3R}\left(1 + \frac{2}{R^2}\right)Q_0. \qquad (15)$$

The main component of the resistance force is here the pressure resistance. The resultant forces of the components of the viscous stress tensor are exponentially vanishing quantities.

These formulas (14) and (15) for the resistance force in the extreme cases have been published earlier in [3].

The relation $Q/Q_0 = f(R)$ is shown in Fig. 1. A comparison with the Stokes formula indicates that the resistance force decreases appreciably with faster evaporation.

Formula (13) for the resistance force yields the following equation of motion for a droplet evaporating with spherical symmetry in a medium at rest far from the droplet:

$$m\frac{d\mathbf{V}}{dt} = mg - 8\pi\rho\nu a\mathbf{V}\,\frac{R\,[1-(1+R)\,e^{-R}]}{2\,(1+R)\,e^{-R}+R^2-2}, \tag{16}$$

$$m = \frac{4}{3}\pi\rho' a^3,$$

with m denoting the mass of the droplet, V denoting the velocity of the droplet and g denoting the acceleration of free fall.

During spherically symmetrical evaporation of a droplet, the mean-over-its-surface velocity of particles leaving the droplet is equal to the velocity of the droplet.

In this case, according to the Meshcherskii equation [4], the additional term **V**dm/dt does not appear in the equation of motion.

If the droplet particles would leave the droplet surface at the velocity of the ambient medium, then the term **V**dm/dt would have to be added, in accordance with the Meshcherskii equation. In view of this, we note that the equations of motion given in [5] for an evaporating droplet are erroneous.

Indeed, the momentum of the entire system, including the vapor and the droplet, at an instant of time t is

$$m\mathbf{V} + \int \rho\mathbf{v}s\,d^3r,$$

$$s = \theta\,(|\mathbf{r}-\mathbf{R}|-a),\ \theta\,(x) = \begin{cases} 1, & x>0, \\ 0, & x<0, \end{cases}$$

with **R** denoting the radius vector of the center of mass of the droplet.

A change in momentum of the system is equal to the principal vector of external forces acting on the system, which in this case of a droplet amounts to the force of gravity

$$\frac{d}{dt}\,m\mathbf{V} + \frac{d}{dt}\int \rho\mathbf{v}s\,d^3r = mg. \tag{17}$$

The flow of vapor is described by the Navier–Stokes equations and the continuity equation, according to which

$$\frac{d}{dt}\int \rho v_\alpha s\,d^3r = \int \left(-pn_\alpha + \sigma'_{\alpha\beta}\,n_\beta\right) dS - \int \rho v_\alpha\,\left(v_\beta n_\beta - V_\beta n_\beta - \dot{a}\right) dS,$$

with **n** denoting the unit vector normal to surface element dS of the droplet. Here and subsequently, a repetition of subscripts signifies a summation.

By virtue of the continuity of mass flow through the droplet surface

$$\rho\,(v_\beta n_\beta - V_\beta n_\beta - \dot{a}) = -\rho'\dot{a},$$

and the last integral can be expressed as

$$\rho'\dot{a}\int \mathbf{v}\,dS = -\frac{dm}{dt}\,\mathbf{V}.$$

Thus, for spherically symmetrical evaporation of the droplet, from Eq. (17) with formulas (12) and (13) follows Eq. (16).

From the calculated velocity field one can obtain the concentration or the temperature distribution around an evaporating droplet, in terms of the well known equation of convective diffusion. A solution of this problem for $Pe \ll 1$ and $PPe \ll 1$ leads to the following expression for the mean-over-the-surface Nusselt number for a sphere

$$\mathrm{Nu} = \frac{2Pe^{-P}}{1 - e^{-P}} \left[1 + \frac{P\mathrm{Pe}}{2(1 - e^{-P})} \right], \qquad (18)$$

with P and Pe denoting respectively either wa/D and ua/D or wa/χ and ua/χ (here D is the diffusivity and χ is the thermal diffusivity).

When $P \ll 1$ and $\mathrm{Pe} \ll 1$, then accurately within P^2-order terms (including PPe),

$$\mathrm{Nu} = 2 - P + \frac{1}{6} P^2 + \mathrm{Pe}.$$

Comparing this expression with the one obtained earlier in [6, 7] and given here with the same accuracy

$$\mathrm{Nu} = 2 - P + \frac{1}{6} P^2 + \mathrm{Pe} - \frac{1}{4} P\mathrm{Pe},$$

we note that there is no PPe-order term in the former. This is a result of erroneous choice of the flow function in [6, 7] as a superposition of a point source on the Stokes solution rather than on the solution to the appropriately linearized Navier–Stokes equation (4) with convective terms representing the radial flow of vapor. The expression for the flow function used in [6, 7] is suitable if only the first terms of the Nu-number expansion in terms of parameters $P \ll 1$ and $\mathrm{Pe} \ll 1$ are considered, i. e., suitable for the calculation of heat and mass transfer from a slowly evaporating droplet in a viscous stream, while formula (18) covers a wider range which includes a fast evaporating droplet ($P \gg 1$) in a slow stream ($\mathrm{Pe} \ll 1/P$).

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